

Homework 8

Math 121A, Fall 2025.

Due: Wednesday, Oct 22, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. Division of power series is often more conveniently handled by appropriately using asymptotic notation and known expansions, rather than by solving linear equations or performing long division. As an example, observe that

$$\frac{1}{\sin z} = \frac{1}{z - z^3/3! + O(z^5)} = \frac{1}{z} \left(\frac{1}{1 - z^2/3! + O(z^4)} \right)$$

Since $z^2/3! + O(z^4)$ tends to 0 as $z \rightarrow 0$, we have in particular $|z^2/3! + O(z^4)| < 1$ for sufficiently small z (which is all we are interested in). Thus, the geometric expansion is valid in a neighborhood of 0, and we get

$$\frac{1}{\sin z} = \frac{1}{z} \left(1 + (z^2/3! + O(z^4)) + O(z^4) \right) = \frac{1}{z} + \frac{z}{3!} + O(z^3),$$

which is consistent with the result obtained by long division. Use the above method to find the first few terms of the Laurent series about 0 for $\frac{1}{e^z-1}$ and $\cot(z)$.

Problem 2. Find the principal parts of the Laurent expansions about 0 of the following functions, indicating the type of singularity in each case.

- (a) $\frac{1}{\sin^2(z)}$.
- (b) $\frac{e^z+1}{e^z-1}$.

Problem 3. Find the first few terms of the Laurent series for each of the following functions around the indicated points so that you may compute the residue.

- (a) $\frac{e^z}{z^2-1}, z = 1$
- (b) $\sin \frac{1}{z}, z = 0$

Problem 4. (a) Problem 14.4.3
(b) Problem 14.4.7

Problem 5. Find the residues of the following functions at the indicated points:

- (a) $\frac{e^z-1}{\sin z}, z_0 = 0$.
- (b) $\frac{e^z}{(z^2-1)^2}, z_0 = 1$.
- (c) $\left(\frac{\cos z-1}{z} \right)^2, z_0 = 0$.

(d) $\frac{z+2}{z^2-2z}, z_0 = 0.$

Problem 6. Give examples of functions f_1, f_2 with the following properties:

- (a) f_1 has a simple pole at $z = 0$ and an essential singularity at $z = 1$.
- (b) f_2 has a removable singularity at $z = 0$, a pole of order 6 at $z = 1$, and an essential singularity at $z = i$.

Problem 7. Locate and classify (as removable, essential, or poles of a specified order) the singularities of the following functions:

- (a) $\frac{1}{(\pi+z)\sin(z)}.$
- (b) $\frac{1}{(z^2+z+1)^3}.$
- (c) $\sin(1 - 1/z).$
- (d) $\csc(\sin(z)).$

Problem 8. Suppose that f is entire (i.e. holomorphic on all of $\mathbb{C}$) and $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Prove that f is constant.

Problem 9. Let f be entire and suppose there exists a real M such that $|f(z)| \leq M|z|$. Prove that f is linear, i.e. $f(z) = az$ for some $a \in \mathbb{C}$.

Problem 10. (a) Use Cauchy's integral formula to show that if f is holomorphic on and in the circle of radius r centered at z_0 , then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

This is called the *mean value property*.

- (b) Let $u : \mathbb{C} \rightarrow \mathbb{R}$ be given by $u(z) = \operatorname{Re}(f(z))$ (recall from last homework that this means u is harmonic). Extract a mean value property for u from part (a). [There is a theorem which guarantees that harmonic functions \$u\$ on a disc are always the real part of a holomorphic function. This means that solutions to Laplace's equation \$\Delta u = 0\$ on the disc always satisfy the mean value property!](#)